

KHOVANOV-ROZANSKY HOMOLOGY AND CONWAY MUTATION

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ABSTRACT. We show that the reduced $\mathfrak{sl}(n)$ homology defined by Khovanov and Rozansky is invariant under component-preserving positive mutation when n is odd.

1. INTRODUCTION

Understanding the behavior of Khovanov homology under Conway mutation has been an active area of study. Wehrli [26] demonstrated that unlike the Jones polynomial, Khovanov homology detects mutation of links. Bar-Natan [2] showed that for a pair of mutant knots (or, more generally, two links that are related by component-preserving mutation) there are two spectral sequences with identical E_2 pages converging to the Khovanov homologies of the knots. Champanerker and Kofman [5] relate Khovanov homology to a (mutation-invariant) matroid obtained from the Tait graph of a knot diagram. The question remains open, but with coefficients in \mathbb{Z}_2 it was solved independently by Bloom [4] and Wehrli [27]. In fact, Bloom proves the more general result that odd Khovanov homology (see Ozsváth, Rasmussen and Szabó [22]) is invariant under arbitrary mutation of links. A similar statement cannot hold for the original Khovanov homology, as we know from Wehrli's example in [26]. Recently, Kronheimer, Mrowka and Ruberman [8] showed that the total rank of instanton knot homology is invariant under genus-2 mutation, which implies invariance under Conway mutation.

In this paper, we investigate the effect of mutation on $\mathfrak{sl}(n)$ homology, a generalization of Khovanov homology. $\mathfrak{sl}(n)$ homology, defined by Khovanov and Rozansky in [12], is a categorification of the $\mathfrak{sl}(n)$ polynomial, a certain specialization of the HOMFLY-PT polynomial that can be obtained from the fundamental n -dimensional representation of $U_q(\mathfrak{sl}(n))$. As noted implicitly by Gornik [7] and later used by Rasmussen in [23] (see also Krasner [10] and Wu [29]), the definitions make sense in a more general context: To any polynomial $p \in \mathbb{Q}[x]$, one can assign a homology theory that conjecturally only depends on the multiplicities of the (complex) roots of $p'(x)$. $\mathfrak{sl}(n)$ homology is recovered by setting $p(x) = \frac{1}{n+1}x^{n+1}$. For odd n , we establish invariance under positive mutation, that is mutation that respects the orientations of both 2-tangles involved in it.

Theorem 1.1. *If L and L' are two links related by component-preserving positive mutation and n is odd, then their reduced $\mathfrak{sl}(n)$ homologies are isomorphic (reduced with respect to the component of the mutation, to be defined in [section 2](#)). More generally, let $p(x) = \sum_k a_{2k}x^{2k}$ be a polynomial with only even powers of x , then the reduced Khovanov-Rozansky homologies of L and L' associated to this polynomial are isomorphic.*

Using Rasmussen's spectral sequence from HOMFLY-PT homology to $\mathfrak{sl}(n)$ homology, and the fact that HOMFLY-PT homology of knots is finite dimensional, we get the following

Corollary 1.2. *If K and K' are two knots related by positive mutation, then their HOMFLY-PT homologies are isomorphic.*

We prove the theorem by first showing that the Khovanov-Rozansky complex of the inner 2-tangle can be built out of the complexes assigned to two basic diagrams: a pair of arcs and a singular crossing. For the Khovanov-Rozansky complex, we follow Rasmussen's definitions from [23], since Khovanov and Rozansky's original definitions are not general enough to serve our purpose. We then derive a criterion for a certain mapping cone of this complex to be invariant under reflection, which turns out to be the case for odd n in the case of positive mutation. Closing up the tangle, we see that the mapping cone computes reduced Khovanov-Rozansky homology.

The methods used in the proof are fairly general, and we envision that they can be used to show mutation invariance in other contexts.

- They apply to (unreduced) \mathbb{Z}_2 Khovanov homology and more generally to the equivariant theory over the ring $\mathbb{Z}_2[h, t]$ defined by Khovanov in [15]. In both cases we get mutation invariance under arbitrary component-preserving mutation, regardless of whether mutation is positive or negative.
- They also apply to Khovanov's (integral) $\mathfrak{sl}(3)$ homology and Mackaay and Vaz's corresponding equivariant theory over the ring $\mathbb{Z}[a, b, c]$ [19]. We plan to show in a later paper that the corresponding theory over $\mathbb{Z}_2[a, b, c]$ is invariant under component-preserving mutation and that an appropriately defined reduced theory over $\mathbb{Z}[b]$ is invariant under positive component-preserving mutation.
- We expect that analogs of the previous statements hold for larger n : For odd n , a reduced version of Krasner's equivariant $\mathfrak{sl}(n)$ homology [10] should be invariant under positive component-preserving mutation when setting the variables corresponding to coefficients of odd powers of p to 0. For arbitrary n , we expect invariance under component-preserving mutation when working with coefficients in \mathbb{Z}_2 . Note, however, that the standard definition of $\mathfrak{sl}(n)$ homology only works with coefficients in \mathbb{Q} . Krasner [11] proposed a definition of an integral version of $\mathfrak{sl}(n)$ homology.
- Equivariant versions of $\mathfrak{sl}(n)$ homology give rise to spectral sequences that can be used to define analogs of Rasmussen's (integer-valued) s -invariant [24] for $\mathfrak{sl}(n)$ homology. Generalized s -invariants have their source in Gornik's work [7] and have been studied by Lobb [18] and Wu [29]. One can define such an invariant for any polynomial over \mathbb{C} of degree n with only single roots, but it is not known whether the invariant depends on this choice of polynomial. We expect that for odd n , they are invariant under positive component-preserving mutation — at least for a particular choice of polynomial. Furthermore, we hope to show invariance of the original s -invariant by considering the equivariant $\mathfrak{sl}(2)$ theory over \mathbb{Z}_2 .
- We also expect that Khovanov and Rozansky's HOMFLY-PT homology [13] is invariant under arbitrary mutation. A different set of technical difficulties arises when studying this question; we hope to return to the question in a future paper.

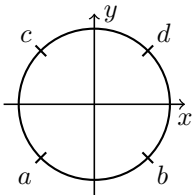


FIGURE 1. Placement and labeling of the endpoints of the inner tangle

- More generally, we expect Rasmussen's spectral sequence from HOMFLY-PT homology to $\mathfrak{sl}(n)$ homology to be invariant for odd n , although it is less clear how to apply our technique since the data the spectral sequence is constructed from lacks a satisfactory equivalent for tangles.
- It will be interesting to see if a generalization of our method can be applied to show invariance of \mathbb{Z}_2 Khovanov homology under genus-2 mutation [6].

Finally, we note that our result is consistent with calculations for the Kinoshita-Terasaka knot and the Conway knot carried out by Mackaay and Vaz [20].

The organization of this paper is as follows. In [section 2](#), we review relevant definitions and explain our conventions. In [section 3](#), we reduce the problem to the case of mutation of a 2-tangle in what we call braid form. In [section 4](#), we investigate how the Khovanov-Rozansky complex behaves under positive mutation. In [section 5](#), we show how to represent the Khovanov-Rozansky complex of a 2-tangle in braid-form as a complex over a particularly simple category. In [section 6](#), we derive a general criterion for when a chain complex over an additive category is isomorphic to its image under a certain involution functor and show how this criterion applies to the problem at hand. In [section 7](#), we combine the results from the previous sections to prove [Theorem 1.1](#).

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2. DEFINITIONS

Conway mutation is the process of decomposing a link L as the union of two 2-tangles $L = T \cup T'$ and then regluing in a certain way. Diagrammatically, we may assume that one of the tangles (the 'inner' tangle T) lies inside a unit circle with endpoints equally spaced as in [Figure 1](#). Mutation consists of one of the following transformations R of the inside tangle, followed by regluing: reflection along the x -axis (R_x), reflection along the y -axis (R_y) or rotation about the origin by 180 degrees (R_z). In other words, the mutant is given by $L' = R(T) \cup T'$. When taking orientations into account, we can distinguish two types of mutation (see for example Kirk and Livingston [9]).

Definition 2.1. Mutation of an oriented link is called *positive* if orientations match when regluing, i.e. if $L' = R(T) \cup T'$ as an oriented link and it is called *negative* if the orientation of the inner tangle needs to be reversed before regluing, i.e. if

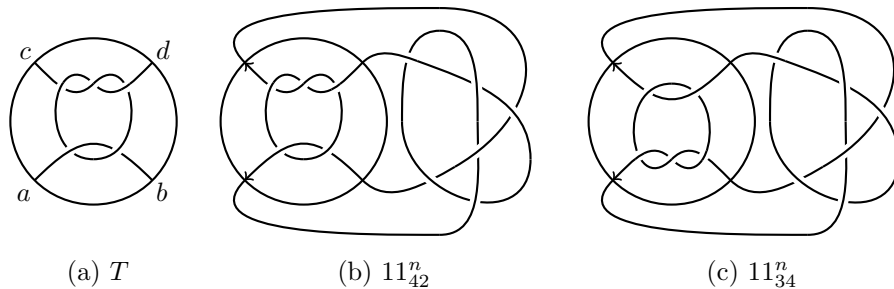


FIGURE 2. A 2-tangle with orientation-reversing symmetry and the Kinoshita-Terasaka - Conway mutant pair

$L' = -R(T) \cup T'$ as an oriented link, where $-R(T)$ denotes $R(T)$ with orientations reversed.

As an example, consider the two knots in Figure 2. 11_{34}^n is a positive mutant of 11_{42}^n , since rotation about the y -axis preserves the orientations of the ends of T . It is also a negative mutant, as can be seen by considering rotation about the z -axis.

There are 16 mutant pairs with 11 or fewer crossings, see [17]. It can be checked that all of them can be realized by negative mutation. Among the 16 pairs, we found 5 that can be realized on the tangle T depicted in Figure 2(a): $(11_{57}^a, 11_{231}^a)$, $(11_{251}^a, 11_{253}^a)$, $(11_{34}^n, 11_{42}^n)$, $(11_{76}^n, 11_{78}^n)$ and $(11_{151}^n, 11_{152}^n)$. $R_y(T)$ is isotopic to T but with orientations reversed, therefore these 5 mutant pairs can be realized by both positive and negative mutation. In particular, our proof applies to the famous Kinoshita-Terasaka - Conway pair, illustrated in Figure 2(b) and (c).

Definition 2.2. Mutation of a link is called *component-preserving* if a and $R(a)$ lie on the same component of the original link (or equivalently, on the same component of the mutant).

Note that knot mutation is always component-preserving. If positive mutation is component-preserving, then a and $R(a)$ are either both incoming or both outgoing edges, hence all 4 endpoints lie on the same component of the link. We referred to this component earlier as the *component of the mutation*.

For Khovanov-Rozansky homology, our definitions closely follow [23], but note that we work with \mathbb{Z}_2 -graded matrix factorizations instead of \mathbb{Z} -graded ones in order to get a stronger version of invariance under Reidemeister moves.

A matrix factorization over a commutative ring R with potential $w \in R$ is a free \mathbb{Z}_2 -graded module C^* equipped with a differential $d = (d_0, d_1)$ such that $d^2 = w \cdot I_C$. Following [23], we use the notation

$$C^1 \xrightleftharpoons[d^0]{d^1} C^0$$

Morphisms are simply degree-0 maps between matrix factorizations which commute with the differential. We denote the category of matrix factorizations over R with potential w by $MF_w(R)$. We say that two morphisms of matrix factorizations $\phi, \psi : C \rightarrow C'$ are homotopic if $\phi - \psi = d_{C'}h + hd_C$ for some degree-1 homotopy $h : C \rightarrow C'$. The category of matrix factorizations over R with potential w and

morphisms considered up to homotopy will be denoted by $HMF_w(R)$. For a graded ring R , whose grading we will call q -grading, we also introduce a notion of graded matrix factorizations with homogeneous potential w by requiring that both d^0 and d^1 be homogeneous of q -degree $\frac{1}{2} \deg w$. Morphisms between graded matrix factorization are required to have q -degree 0, whereas homotopies must have q -degree $-\frac{1}{2} \deg w$. The corresponding homotopy category of graded matrix factorizations will be denoted by $hmf_w(R)$. For the three different gradings in $hmf_w(R)$ we introduce three types of grading shifts: A shift in the \mathbb{Z}_2 grading coming from matrix factorizations will be denoted by $\langle \cdot \rangle$, a shift in homological grading by $[\cdot]$ and a shift in q -grading by $\{\cdot\}$. We follow the convention that $R[n]$ has a single generator in homological height n , and similarly for $\{\cdot\}$. Note that if $\phi : A \rightarrow B$ has q -degree d , then the q -degree of $\phi : A\{k_A\} \rightarrow B\{k_B\}$ is $d + k_B - k_A$.

An important class of matrix factorizations is the class of Koszul factorizations, which we will briefly describe here. For a more detailed treatment, we refer the reader to Section 2.2 of [14] (but note that we switched the order of the arguments of K in order to be consistent with [13] and [23]). If $u, v \in R$, then $K(u; v)$ is the factorization

$$R\left\{\frac{\deg v - \deg u}{2}\right\} \xrightleftharpoons[u]{v} R$$

We will sometimes write $K_R(u; v)$ to clarify which ring we are working over. For $\mathbf{u} = (u_1, \dots, u_n)^T$, $\mathbf{v} = (v_1, \dots, v_n)^T$ we define $K(\mathbf{u}, \mathbf{v}) = \bigotimes_{k=1}^n K(u_k; v_k)$. This is a matrix factorization with potential $\sum_{k=1}^n a_k b_k$. We will also use the notation

$$K(\mathbf{u}, \mathbf{v}) = \begin{Bmatrix} u_1 & v_1 \\ \vdots & \vdots \\ u_n & v_n \end{Bmatrix}$$

If we are not interested in \mathbf{u} , we may apply arbitrary row transformations to \mathbf{v} : for an invertible matrix X , $K(\mathbf{u}, \mathbf{v}) \cong K((X^{-1})^t \mathbf{u}, X \mathbf{v})$. We describe order-two Koszul matrix factorizations explicitly, thereby fixing a sign convention for the tensor product of matrix factorizations:

$$\begin{Bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{Bmatrix} = R\{k_1\} \oplus R\{k_2\} \xrightleftharpoons[\begin{pmatrix} v_2 & u_1 \\ v_1 & -u_2 \end{pmatrix}]{\begin{pmatrix} u_2 & u_1 \\ v_1 & -v_2 \end{pmatrix}} R\{k_1 + k_2\} \oplus R$$

Here $k_1 = \frac{\deg v_1 - \deg u_1}{2} = \deg v_1 - \frac{\deg w}{2}$ and $k_2 = \frac{\deg v_2 - \deg u_2}{2} = \deg v_2 - \frac{\deg w}{2}$.

Our definition of Khovanov-Rozansky homology closely follows Rasmussen [23], whose definitions we amend slightly for technical reasons. We also restrict ourselves to connected diagrams. To any diagram of a (possibly singular) oriented tangle, which we allow to contain any of the diagrams depicted in Figure 3 as subdiagrams, Rasmussen defines two rings, which depend only on the underlying 4-valent graph obtained by replacing all of those diagrams by a vertex. The *edge ring* $R(D)$ is the polynomial ring over \mathbb{Q} generated by variables x_e , where e runs over all edges of the diagram, subject to a relation of the form $x_a + x_b - x_c - x_d$ for each vertex of the underlying 4-valent graph. By setting $\deg x_e = 2$ for each edge e of D , $R(D)$ becomes a graded ring. The *external ring* $R_e(D)$ is the subring of $R(D)$ generated by the variables associated to the endpoints of D . Lemma 2.5 in [23] shows that if

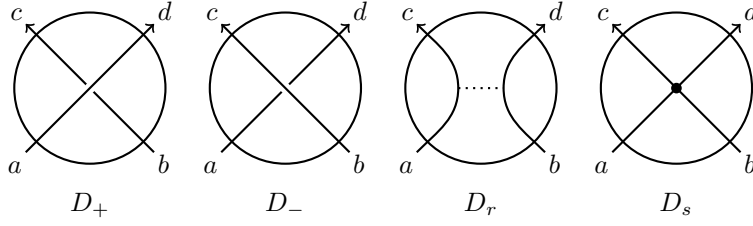


FIGURE 3. Positive crossing, negative crossing, oriented smoothing and singular crossing. The dotted line connecting the two arcs of the oriented smoothing illustrates that we consider both arcs to be on the same component of the smoothing.

we associate the variables x_i ($i \in \{1, 2, \dots, k\}$) to the incoming edges of D and y_i to the outgoing edges, then $R_e(D) \cong \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] / (\sum_i y_i - \sum_i x_i)$.

Fix a polynomial $p \in \mathbb{Q}[x]$. If p is not homogeneous, we will disregard q -gradings below. To each tangle diagram D , we will associate a complex $C_p(D)$ of matrix factorizations over $R(D)$, which we consider to be an object of the category $K^b(hmf_w(R_e(D)))$, where $K^b(\mathcal{C})$ denotes the homotopy category of bounded complexes over the additive category \mathcal{C} and $w = \sum_i p(y_i) - \sum_i p(x_i)$ where x_i and y_i are associated to the incoming and outgoing edges as above. $C_p(D)$ is first defined on the diagrams shown in Figure 3. In each case $R := R(D) = R_e(D) = \mathbb{Q}[x_a, x_b, x_c, x_d] / (x_a + x_b = x_c + x_d)$. We set

$$\begin{aligned} C_p(D_r) &= K(x_c - x_a; *) \langle 1 \rangle = K(*; x_c - x_a) \{n-1\}, \\ C_p(D_s) &= K(*; x_c x_d - x_a x_b) \{-1\}, \\ C_p(D_+) &= K(*; x_c x_d - x_a x_b) [-1] \xrightarrow{d_+} K(*; x_c - x_a) \text{ and} \\ C_p(D_-) &= K(*; x_c - x_a) \xrightarrow{d_-} K(*; x_c x_d - x_a x_b) \{-2\} [1]. \end{aligned}$$

Here $*$ is of course determined by the potential in each case; we postpone the definitions of d_+ and d_- until we need them in Lemma 4.1.

This definition is extended to arbitrary tangle diagrams by the formula

$$(1) \quad C_p(D) = \bigotimes_i C_p(D_i) \otimes_{R(D_i)} R(D),$$

where D_i runs over all crossings of D and the big tensor product is taken over $R(D)$. As indicated above, we usually view $C_p(D)$ as a matrix factorization over the smaller ring $R_e(D)$.

Rasmussen shows (Lemma 2.8 in [23])

Proposition 2.3. *If D is obtained from D_1 and D_2 by taking their disjoint union and identifying external edges labeled x_1, \dots, x_k in both diagrams, then*

$$\begin{aligned} R(D) &\cong R(D_1) \otimes_{\mathbb{Q}[x_1, \dots, x_k]} R(D_2) \text{ and} \\ C_p(D) &\cong C_p(D_1) \otimes_{\mathbb{Q}[x_1, \dots, x_k]} C_p(D_2). \end{aligned}$$

To define reduced Khovanov-Rozansky homology of a link with respect to a marked component, we pick an edge on the marked component, which we label by x . We view $C_p(D)$ as a matrix factorization over $\mathbb{Q}[x]$, i.e. as an object of $K^b(hmf_0(\mathbb{Q}[x]))$. Alternatively, we may consider the diagram D° obtained from D

by cutting it open at the marked edge. Let x and y be the labels of the incoming and outgoing edge of D° , respectively. Then $C_p(D^\circ)$ is a complex of matrix factorization with potential $p(y) - p(x) = 0$ over the ring $R_e(D^\circ) = \mathbb{Q}[x, y]/(y - x) \cong \mathbb{Q}[x]$ and $C_p(D) \cong C_p(D^\circ)$ as objects of $K^b(hmf_0(\mathbb{Q}[x]))$.

We define reduced Khovanov-Rozansky homology in two steps. We first define the unreduced complex $\tilde{C}_p(D^\circ)$ by tensoring with $K(p'(x); 0)$. Then we define the reduced complex $\hat{C}_p(D^\circ)$ as the mapping cone $\text{Cone}(x : C_p(\tilde{D}^\circ)\{2\} \rightarrow C_p(\tilde{D}^\circ))$ (we use $\hat{\cdot}$ rather than $\bar{\cdot}$ in order to avoid confusion with the involution $\bar{\cdot}$ to be defined later).

We now consider $\hat{C}_p(D^\circ)$ as an object of $K^b(hmf_0(\mathbb{Q}))$. Since we are working over a field and matrix factorizations with potential 0 are simply \mathbb{Z}_2 -graded chain complexes, the category $hmf_0(\mathbb{Q})$ is equivalent to the category of $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded \mathbb{Q} -vector spaces by [Proposition 2.7](#) below. Hence the category $K^b(hmf_0(\mathbb{Q}))$ is equivalent to the category $K^b(\mathbb{Z}_2 \oplus \mathbb{Z}\text{-graded } \mathbb{Q}\text{-vector spaces})$, which in turn is equivalent to the category of $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$ -graded \mathbb{Q} -vector spaces (bounded with respect to the second \mathbb{Z} summand) by [Proposition 2.6](#). Reduced Khovanov-Rozansky homology is the image of $\hat{C}_p(D^\circ)\{(n-1)w\}$ under this equivalence of categories, where w is the writhe of D .

Proposition 2.4. *The definition of reduced $\mathfrak{sl}(n)$ homology above is equivalent to Khovanov and Rozansky's original definition in [12].*

Proof. Rasmussen did most of the work for us in his proof of Proposition 3.12 in [23]. Using the notation d_{tot} for the (inner) matrix factorization differential and d_v for the (outer) differential of a complex of matrix factorizations, he shows in Lemma 3.11 that the original definition in [12] is equivalent to $H^*(H/(x))$, where H is the chain complex whose underlying \mathbb{Q} -vector space is $H^*(C_p(D); d_{tot})$ and whose differential is d_v^* . Our definition above is equivalent to $H^*(\text{Cone}(x : H \rightarrow H))$. Since $\text{Cone}(x : H \rightarrow H)$ has a natural double complex structure, it induces a spectral sequence that converges to $H^*(\text{Cone}(x : H \rightarrow H))$. The E^1 page of the spectral sequence is $H/(x)$, so the E^2 page is $H^*(H/(x))$. The fact that the spectral sequence collapses at the E^2 page implies that the two definitions are equivalent. It is easy to see that gradings match as well, as Rasmussen explains at the end of his proof of Proposition 3.12. \square

Remark 2.5. We could have more straightforwardly defined reduced homology as the homology of $\text{Cone}(x : C_p(D) \rightarrow C_p(D))$. However, this would have required us to show that $C_p(D)$ is torsion-free as a $\mathbb{Q}[x]$ -module in order to establish equivalence of definitions, which follows from [Proposition 5.5](#) below only if D is a braid diagram.

The following two propositions are well-known in the finitely generated case. We verify that proof carries over to the infinitely generated setting.

Proposition 2.6. *Let $C = \dots \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \dots$ be a chain complex over \mathbb{Q} (with not necessarily finitely generated chain groups). Then C is homotopy equivalent to a complex with zero differential (its cohomology).*

Proof. As usual, let $Z^k := \ker(d^k)$ and $B^{k+1} := \text{im}(d^k)$. Since vector spaces are free as modules, the short exact sequences $0 \rightarrow Z^k \rightarrow C^k \xrightarrow{d^k} B^{k+1} \rightarrow 0$ and $0 \rightarrow B^k \rightarrow Z^k \rightarrow H^k(C) \rightarrow 0$ split. It is easy to check that with respect to the decomposition $C^k \cong Z^k \oplus B^{k+1} \cong B^k \oplus H^k(C) \oplus B^{k+1}$, C decomposes as a direct

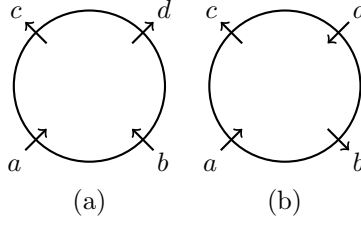


FIGURE 4. The two possible orientations of the endpoints of inner tangle

sum of chain complexes $0 \rightarrow H^k(C) \rightarrow 0$ and $0 \rightarrow B^k \xrightarrow{\text{id}} B^k \rightarrow 0$. The Proposition now follows from the fact that the latter chain complex is homotopy equivalent to the zero complex. \square

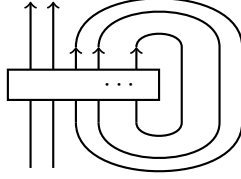
Proposition 2.7. *Any \mathbb{Z}_2 -graded chain complex $C^1 \xrightleftharpoons[d^0]{d^1} C^0$ is homotopy equivalent to a \mathbb{Z}_2 -graded chain complex with zero differential.*

Proof. Arguing as in the proof of the previous Proposition, we may decompose C as a direct sum of $H^1(C) \xrightleftharpoons{\quad} 0$, $0 \xrightleftharpoons{\quad} H^0(C)$, $B^1 \xrightleftharpoons[\text{id}]{0} B^1$ and $B^0 \xrightleftharpoons[0]{\text{id}} B^0$, where the latter two complexes are homotopy equivalent to zero complexes. \square

3. TOPOLOGICAL CONSIDERATIONS

In this section, we show that we may assume that the inner tangle is presented in a specific form.

Definition 3.1. We say that a 2-tangle is in braid form if it is represented in the following way, where the rectangle represents an open braid.



Theorem 3.2. *Let L be an oriented link and L' be a mutant of L obtained by positive mutation. Then the mutation can be represented on a diagram whose inner tangle is given in braid form by a transformation of type R_y .*

The following two lemmas immediately imply the Theorem.

Lemma 3.3. *We may assume that the endpoints of the inner tangle are oriented as in Figure 4(a) and that the transformation of the inner tangle is of type R_y .*

Lemma 3.4. *Any 2-tangle with endpoints oriented as in Figure 4(a) can be represented by a diagram in braid form.*

Proof (of Lemma 3.3). If the tangle has two adjacent endpoints with the same orientation, it is isotopic to a tangle with endpoints as depicted in Figure 4(a) and the only positive mutation is of type R_y . Otherwise we are in case (b) of Figure 4,

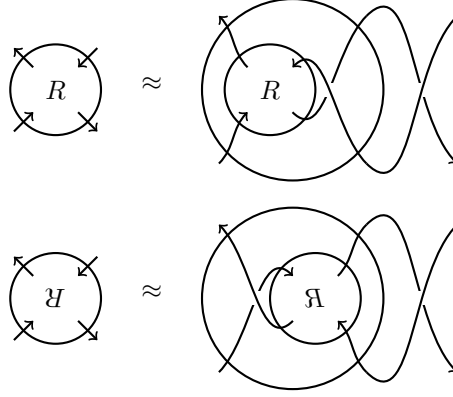


FIGURE 5. R_z mutation on a tangle of type (b) is equivalent to R_y mutation on a tangle of type (a)

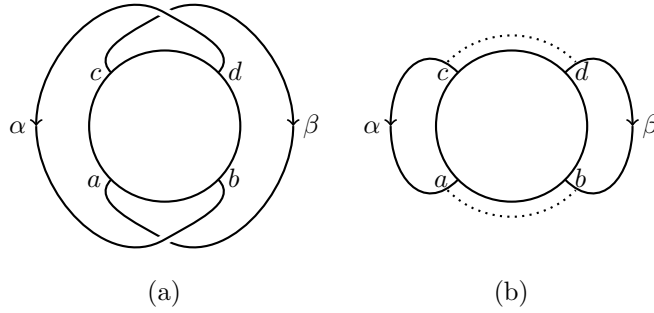


FIGURE 6. A closure of the tangle and its Seifert picture

where the only positive mutation is of type R_z . But we can realize this type of mutation by R_y -mutation on a tangle of type (a), as illustrated in Figure 5. \square

Proof (of Lemma 3.4). The proof uses a slight variation of the Yamada-Vogel [25, 30] algorithm to prove an analog of Alexander's Theorem for 2-tangles. We follow Birman and Brendle [3].

Close the tangle by two arcs α from c to a and β from d to b as in Figure 6(a). The algorithm works by repeatedly performing a Reidemeister II move in a small neighborhood of a so-called *reducing arc*. The algorithm is performed on the Seifert picture of the link diagram, which is depicted in Figure 6(b). A reducing arc is an arc connecting an incoherently oriented pair of Seifert circles that intersects the Seifert picture only at its endpoints. Since the Seifert circles that α and β belong to are coherently oriented, the unbounded region of the Seifert picture in Figure 6(b) cannot contain a reducing arc. Hence we may push the reducing arc into the circle. The algorithm now gives us a tangle diagram whose Seifert circles and Seifert arcs (from a to c and from b to d) are coherently oriented. This implies that all Seifert circles lie nested inside each other to the left of the left arc and to the right of the right arc, in other words it can be represented by a diagram in the

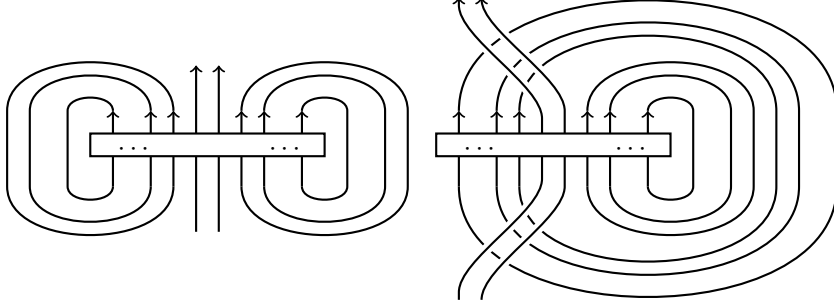


FIGURE 7. Transforming the diagram into braid form

form illustrated on the left of Figure 7. But this can be easily transformed into braid form, as seen on the right of Figure 7. \square

4. BEHAVIOR OF THE KHOVANOV-ROZANSKY CHAIN COMPLEX UNDER REFLECTION

Lemma 4.1. *Let D be an oriented (possibly singular) tangle diagram and \bar{D} be the reflection of D . Label the endpoints of D by $e_0, e_1, \dots, e_{2k-1}$, and the corresponding endpoints of \bar{D} by $e'_0, e'_1, \dots, e'_{2k-1}$. Then $C_p(\bar{D}) = \phi(C_p(D))$, where $\phi : R(D) \rightarrow R(\bar{D})$ is the ring homomorphism given by $\phi(x_{e_i}) = -x_{e'_i}$.*

Proof. If D is one of the diagrams shown in Figure 3, $C_p(D)$ is one of the following complexes of matrix factorizations.

$$\begin{array}{ccc}
 C_p(D_+): & & C_p(D_-): \\
 \begin{array}{ccc}
 R\{1-n\} & \xrightleftharpoons[\frac{w}{x_c-x_a}]^{x_c-x_a} & R \\
 \uparrow x_c-x_b & & \uparrow 1 \\
 R\{3-n\} & \xrightleftharpoons[\frac{w}{(x_c-x_a)(x_c-x_b)}}{(x_c-x_a)(x_c-x_b)} & R
 \end{array} & & \begin{array}{ccc}
 R\{1-n\} & \xrightleftharpoons[\frac{w}{(x_c-x_a)(x_c-x_b)}}{(x_c-x_a)(x_c-x_b)} & R\{-2\} \\
 \uparrow 1 & & \uparrow x_c-x_b \\
 R\{1-n\} & \xrightleftharpoons[\frac{w}{x_c-x_a}]^{x_c-x_a} & R
 \end{array} \\
 C_p(D_r): & & C_p(D_s): \\
 R & \xrightleftharpoons[\frac{w}{x_c-x_a}]^{x_c-x_a} & R\{n-1\} & & R\{2-n\} & \xrightleftharpoons[\frac{w}{(x_c-x_a)(x_c-x_b)}}{(x_c-x_a)(x_c-x_b)} & R\{-1\}
 \end{array}$$

Note that $\phi(x_c - x_a) = -x_{d'} + x_{b'} = x_{c'} - x_{a'}$, $\phi(x_c - x_b) = -x_{d'} + x_{a'} = x_{c'} - x_{b'}$ and $\phi(w) = \phi(\sum_k a_{2k}(x_c^{2k} + x_d^{2k} - x_a^{2k} - x_b^{2k})) = \sum_k a_{2k}(x_{c'}^{2k} + x_{d'}^{2k} - x_{a'}^{2k} - x_{b'}^{2k}) = w'$. Hence all maps in the above diagrams are mapped by ϕ to the same maps with $x_{a'}$, $x_{b'}$, $x_{c'}$ and $x_{d'}$ in place of x_a , x_b , x_c and x_d , respectively, that is ϕ maps $C_p(D)$ to $C_p(\bar{D})$.

The general case follows from (1): It is clear that by taking the internal edges of D into consideration, we can extend ϕ to an isomorphism between $R(D)$ and $R(\bar{D})$.

Hence we get isomorphisms $C_p(D_i) \otimes_{R(D_i)} R(D) \cong C_p(\bar{D}_i) \otimes_{R(\bar{D}_i)} R(\bar{D})$, which in turn induce an isomorphism $C_p(D) \cong C_p(\bar{D})$. \square

In light of the Lemma, we will simply denote the homomorphism ϕ by $\bar{\cdot}$.

5. KHOVANOV-ROZANSKY HOMOLOGY OF 2-TANGLES

In this section, we investigate the Khovanov-Rozansky homology of 2-tangles in braid form. Denote the variables corresponding to the endpoints a, b, c and d of the tangle by x_a, x_b, x_c and x_d , respectively. The complex associated to such a tangle is a complex of graded matrix factorizations over the ring $R = \mathbb{Q}[x_a, x_b, x_c, x_d]/(x_a + x_b = x_c + x_d)$ with potential $w = p(x_c) + p(x_d) - p(x_a) - p(x_b)$. Let hmf_2 denote the full subcategory of $hmf_w(R)$ whose objects are direct sums of shifts of $C_p(D_r)$ and $C_p(D_s)$.

Theorem 5.1. *Let D a connected diagram of a 2-tangle in braid-form. Then $C_p(D)$ is isomorphic in $K^b(hmf_w(R))$ to an object of $K^b(hmf_2)$.*

Before proving the theorem, we need to recall an important tool for dealing with matrix factorizations: ‘excluding a variable’. We quote Theorem 2.2 from [14].

Theorem 5.2. *Let R be a graded polynomial ring over \mathbb{Q} and $u, v \in R[y]$ two polynomials, with b being monic in y . Furthermore, let $\bar{w} \in R$ and M be a graded matrix factorization over $R[y]$ with potential $w = \bar{w} - uv$. Then $M/(v)$ and $K(u; v) \otimes M$ are isomorphic as objects of $hmf_w(R)$. We say that we exclude the variable y to obtain $M/(v)$ from $K(u; v) \otimes M$.*

The Theorem is only stated for ungraded matrix factorization in [14], but it is trivial to check that the quotient map $K(u; v) \otimes M \rightarrow M/(v)$ constructed in the proof is of degree 0. \square

We will also need another well-known result about Koszul matrix factorizations; this is, for example, the $n = 2$ special case of Theorem 2.1 in [14].

Theorem 5.3. *Let R be a graded polynomial ring over \mathbb{Q} and $v_1, v_2 \in R$ be relatively prime. Then any two Koszul matrix factorizations of the form $\begin{Bmatrix} * & v_1 \\ * & v_2 \end{Bmatrix}$ with the same potential are isomorphic.*

In the same spirit, we show that a matrix factorization that is almost the direct sum of two order-two Koszul matrix factorizations can be transformed into an honest direct sum.

Theorem 5.4. *Let R be a graded polynomial ring and $\dot{R} = R\{\dot{k}\}$ and $\ddot{R} = R\{\ddot{k}\}$ be free R -modules of rank 1, then any graded matrix factorization of the form*

$$\dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} \xrightleftharpoons[U]{V} \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R}$$

with

$$U = \begin{pmatrix} b & * & 0 & * \\ a & * & 0 & * \\ 0 & * & d & * \\ 0 & * & c & * \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} * & * & * & * \\ a & -b & 0 & 0 \\ * & * & * & * \\ 0 & 0 & c & -d \end{pmatrix},$$

$$\left\{ \begin{matrix} * & a \\ * & b \end{matrix} \right\} \{ \dot{k} \} \oplus \left\{ \begin{matrix} * & c \\ * & d \end{matrix} \right\} \{ \ddot{k} \}$$
$$U = \begin{pmatrix} b & * & 0 & b_2 \\ a & * & 0 & a_2 \\ 0 & d_1 & d & * \\ 0 & c_1 & c & * \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} * & * & c_2 & -d_2 \\ a & -b & 0 & 0 \\ a_1 & -b_1 & * & * \\ 0 & 0 & c & -d \end{pmatrix}.$$
$$\begin{array}{ccc} \dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} & \xrightleftharpoons[V]{V} & \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R} \\ \parallel & & \downarrow \scriptstyle X^{-1} \uparrow \scriptstyle X \\ \dot{R}\{k_a\} \oplus \dot{R}\{k_b\} \oplus \ddot{R}\{k_c\} \oplus \ddot{R}\{k_d\} & \xrightleftharpoons[V']{V'} & \dot{R}\{k_a + k_b\} \oplus \dot{R} \oplus \ddot{R}\{k_c + k_d\} \oplus \ddot{R} \end{array}$$
$$X = \begin{pmatrix} 1 & & & \alpha_2 \\ & 1 & & \\ & -\alpha_1 & 1 & \\ & & & 1 \end{pmatrix}, \quad U' = \begin{pmatrix} b & * & 0 & 0 \\ a & * & 0 & 0 \\ 0 & 0 & d & * \\ 0 & 0 & c & * \end{pmatrix} \quad \text{and} \quad V' = \begin{pmatrix} * & * & 0 & 0 \\ a & -b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & c & -d \end{pmatrix},$$

We still need to verify that C is of degree 0: We have $\deg \alpha_1 = \deg a_1 - \deg a = (\frac{\deg w}{2} + k_a + \ddot{k} - k_c - k_d - \dot{k}) - (k_a + \frac{\deg w}{2}) = \ddot{k} - \dot{k} - k_c - k_d$ and $\deg \alpha_2 = \deg a_2 - \deg a = (\frac{\deg w}{2} + \dot{k} - k_b - \ddot{k}) - (k_a + \frac{\deg w}{2}) = \dot{k} - \ddot{k} - k_a - k_b$, which implies $\deg(-\alpha_1 : \dot{R} \rightarrow \ddot{R}\{k_c + k_d\}) = \deg(\alpha_2 : \ddot{R} \rightarrow \dot{R}\{k_a + k_b\}) = 0$. \square

The following proposition is an analog of Lemma 4.10 and Propositions 4.3–4.6 in [23] and Lemma 3 and Propositions 4–7 in [13]. Unfortunately, we cannot deduce it from any of the previous results: The theory introduced in [12] is weaker than what we consider here (in the $\mathfrak{sl}(2)$ case, this is the difference between Khovanov Homology and Bar-Natan’s universal variant [1]). We also cannot use the results in [23], which are only shown to hold up to a notion of quasi-isomorphism. However, the proofs in Rasmussen’s paper can be modified to apply to our situation.

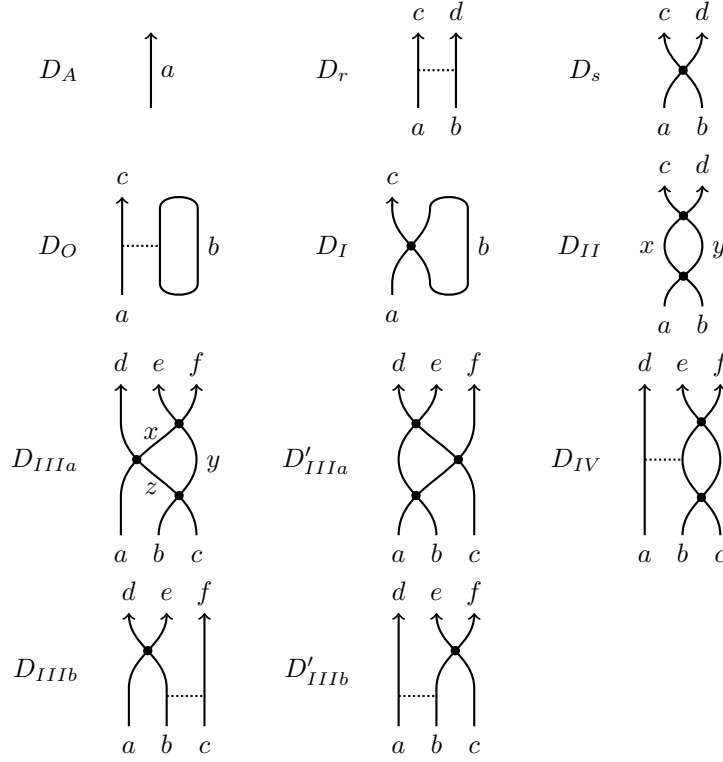


FIGURE 8. Singular braid diagrams

Proposition 5.5. *The following isomorphisms hold in the homotopy category of matrix factorizations over the external ring corresponding to the diagrams.*

- (a) *Let D be a diagram of a fully resolved tangle, and D' be a diagram obtained from D by replacing a smoothing of type D_r (See Figure 3) by a pair of arcs without increasing the number of components. Then $C_p(D) \cong C_p(D')$.*
- (b) *Up to grading shifts, $C_p(D_O)$ is isomorphic to a direct sum of n copies of $C_p(D_A)$.*
- (c) *Up to grading shifts, $C_p(D_I)$ is isomorphic to a direct sum of $n - 1$ copies of $C_p(D_A)$.*
- (d) *Up to grading shifts, $C_p(D_{II}) \cong C_p(D_s) \oplus C_p(D_s)$.*
- (e) *Up to grading shifts, $C_p(D_{IIIa}) \oplus C_p(D_{IIIb}) \cong C_p(D'_{IIIa}) \oplus C_p(D'_{IIIb})$.*
- (f) *Up to grading shifts, $C_p(D_{IV}) \cong C_p(D'_{IIIb}) \oplus C_p(D'_{IIIb})$.*

Proof. (a) Since D and D' are connected, their external rings $R_e(D)$ and $R_e(D')$ are identical. Since $R(D') = R(D)/(x_a = x_c)$ and $R(D) \cong R(D')[x]$ by Lemma 2.4 in [23], x_a and x_c are different elements of $R(D)$. If x_a and x_c were both linear combinations of external edges, then their difference $x_c - x_a$ would be a linear combination of external edges as well. But $x_c - x_a \neq 0 \in R(D)$ and $x_c - x_a = 0 \in R(D')$, which contradicts $R_e(D) = R_e(D')$. Assume w.l.o.g. that x_c is not a linear combination of external edges. Since $K(*; x_c - x_a)$ appears as a factor of $C_p(D)$, we may exclude the variable x_c to obtain $C_p(D')$.

(b) We have $R(D_O) = \mathbb{Q}[x_a, x_b, x_c]/(x_c - x_a)$, $R_e(D_O) = \mathbb{Q}[x_a, x_c]/(x_c - x_a)$ and $R(D_A) = \mathbb{Q}[x_a]$, hence $C_p(D_O) = K\left(x_c - x_a; \frac{p(x_c) + p(x_b) - p(x_a) - p(x_b)}{x_c - x_a}\right) \langle 1 \rangle = K(0; p'(x_a) - p'(x_b)) \langle 1 \rangle$. Excluding the variable x_b , we obtain

$$C_p(D_O) \cong \bigoplus_{i=0}^{n-1} C_p(D_A) \langle 1 \rangle \{2i\}.$$

(c) As in part (b), we have $R(D_s) = \mathbb{Q}[x_a, x_b, x_c]/(x_c - x_a)$, $R_e(D_s) = \mathbb{Q}[x_a, x_c]/(x_c - x_a)$ and $R(D_A) = \mathbb{Q}[x_a]$, so

$$\begin{aligned} C_p(D_I) &= K\left(\frac{p(x_c) + p(x_b) - p(x_a) - p(x_b)}{(x_c - x_a)(x_c - x_b)}; (x_c - x_a)(x_c - x_b)\right) \{-1\} \\ &= K\left(\frac{p'(x_a) - p'(x_b)}{x_a - x_b}; 0\right) \{-1\} \\ &= K\left(0; \frac{p'(x_a) - p'(x_b)}{x_a - x_b}\right) \langle 1 \rangle \{2 - n\}, \end{aligned}$$

so once again we may exclude x_b to get

$$C_p(D_I) \cong \bigoplus_{i=0}^{n-2} C_p(D_A) \langle 1 \rangle \{2 - n + 2i\}.$$

(d) Choose labels as in [Figure 8](#) and set $x := x_x$ and $y := x_y$. As matrix factorizations over $R(D_{II})$,

$$\begin{aligned} C_p(D_{II}) \{2\} &= \begin{Bmatrix} * & (x - x_a)(x - x_b) \\ * & (x_c - x)(x_c - y) \end{Bmatrix} \\ &= \begin{Bmatrix} * & (x - x_a)(x - x_b) \\ * & (x_c - x)(x - x_d) \end{Bmatrix} \\ &\cong \begin{Bmatrix} * & (x - x_a)(x - x_b) \\ * & (x_c - x)(x - x_d) + (x - x_a)(x - x_b) \end{Bmatrix} \\ &= \begin{Bmatrix} * & (x - x_a)(x - x_b) \\ * & (x_c - x_a)(x_c - x_b) \end{Bmatrix} \end{aligned}$$

Let $R = R_e(D_{II}) = R_e(D_s)$. Excluding the variable x , we get a matrix factorization $K(\alpha + \beta x; (x_c - x_a)(x_c - x_b))$ over the ring $R' = R[x]/(x^2 = (x_a + x_b)x - x_a x_b)$ whose potential $(\alpha + \beta x)(x_c - x_a)(x_c - x_b)$ has to lie in R , hence $\beta = 0$. As a graded R -module, $R' \cong R \oplus R\{2\}$, so $C_p(D_{II}) \cong K_R(\alpha; (x_c - x_a)(x_c - x_b)) \{-2\} \oplus K_R(\alpha; (x_c - x_a)(x_c - x_b)) \cong C_p(D_s) \{-1\} \oplus C_p(D_s) \{1\}$.

(e) Choose labels as in [Figure 8](#), and set $x := x_x$, $y := x_y$ and $z := x_z$. Let $R = R_e(D_{IIIa})$, and note that $R(D_{IIIa}) \cong R[x]$. As matrix factorizations over $R(D_{IIIa})$,

$$\begin{aligned} C_p(D_{IIIa}) \{3\} &= \begin{Bmatrix} * & (x_d - x_a)(x_d - z) \\ * & (x_e - x)(x_e - y) \\ * & (z - x_b)(z - x_c) \end{Bmatrix} \\ &= \begin{Bmatrix} * & (x_d - x_a)(x_a - x) \\ * & (x_e - x)(x - x_f) \\ * & (x + x_d - x_a - x_b)(x + x_d - x_a - x_c) \end{Bmatrix} \end{aligned}$$

$$\cong \begin{Bmatrix} * & (x_d - x_a)(x_a - x) \\ * & (x_e - x)(x - x_f) \\ * & x_a x_b + x_b x_c + x_c x_a - x_c x_d - x_d x_e - x_e x_f \end{Bmatrix},$$

where the last line is obtained from the previous one by adding the top right and the center right entry to the bottom right entry. Let $p = x_e + x_f$, $q = x_e x_f$, $\alpha = x_d - x_a$ and $\beta = x_a x_b + x_b x_c + x_c x_a - x_c x_d - x_d x_e - x_e x_f$, so that the last line reads

$$C_p(D_{IIIa})\{3\} \cong \begin{Bmatrix} * & ax_a - ax \\ * & -x^2 + px - q \\ * & b \end{Bmatrix}$$

Using the second row to exclude the variable x , we obtain an order-two Koszul matrix factorization over the ring $R' = R(D_{IIIa})/(x^2 = px - q)$, which is given explicitly (with respect to the standard decomposition of R' as a free R -module of rank two) as $C_p(D_{IIIa}) \cong R_1 \xrightleftharpoons[U]{V} R_0$, where

$$\begin{aligned} R_1 &= R\{3 - n\} \oplus R\{5 - n\} \oplus R\{3 - n\} \oplus R\{5 - n\}, \\ R_0 &= R\{6 - 2n\} \oplus R\{8 - 2n\} \oplus R \oplus R\{2\}, \end{aligned}$$

$$A = \begin{pmatrix} \beta & 0 & * & * \\ 0 & \beta & * & * \\ \alpha x_a & \alpha q & * & * \\ -\alpha & \alpha(x_a - p) & * & * \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ \alpha x_a & \alpha q & -\beta & 0 \\ -\alpha & \alpha(x_a - p) & 0 & -\beta \end{pmatrix}.$$

We apply the following change of basis

$$\begin{array}{ccc} R_1 & \xrightleftharpoons[U]{V} & R_0 \\ X^{-1} \updownarrow X & & X^{-1} \updownarrow X \\ R'_1 & \xrightleftharpoons[U']{V'} & R'_0 \end{array}, \quad \text{where } X = \begin{pmatrix} 1 & p - x_a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x_a \end{pmatrix},$$

$$\begin{aligned} R'_1 &= R\{3 - n\} \oplus R\{5 - n\} \oplus R\{5 - n\} \oplus R\{3 - n\} \text{ and} \\ R'_0 &= R\{6 - 2n\} \oplus R\{2\} \oplus R\{8 - 2n\} \oplus R. \end{aligned}$$

C is of q -degree 0; a straightforward computation shows that

$$\begin{aligned} U' &= \begin{pmatrix} \beta & * & 0 & * \\ -\alpha & * & 0 & * \\ 0 & * & \beta & * \\ 0 & * & \alpha(x_a - x_e)(x_a - x_f) & * \end{pmatrix} \quad \text{and} \\ V' &= \begin{pmatrix} * & * & * & * \\ -\alpha & -\beta & 0 & 0 \\ * & * & * & * \\ 0 & 0 & \alpha(x_a - x_e)(x_a - x_f) & -\beta \end{pmatrix}. \end{aligned}$$

We compute $\gcd(\alpha, \beta) = \gcd(\alpha, (x_d - x_e)\alpha - \beta) = \gcd((x_d - x_a), (x_b - x_e)(x_b - x_f)) = 1$, hence by symmetry $\gcd(x_a - x_e, \beta) = 1$ and $\gcd(x_a - x_f, \beta) = 1$ as

well. Therefore, $\gcd(\alpha, \beta) = \gcd(\alpha(x_a - x_e)(x_a - x_f), \beta) = 1$, so we may apply [Theorem 5.4](#) to get

$$\begin{aligned} C_p(D_{IIIa})\{3\} &\cong \left\{ \begin{smallmatrix} * & -\alpha \\ * & \beta \end{smallmatrix} \right\} \{2\} \oplus \left\{ \begin{smallmatrix} * & \alpha(x_a - x_e)(x_a - x_f) \\ * & \beta \end{smallmatrix} \right\} \\ &\cong \left\{ \begin{smallmatrix} * & -\alpha \\ * & \beta + (x_e + x_f - x_a)\alpha \end{smallmatrix} \right\} \{2\} \oplus \left\{ \begin{smallmatrix} * & \alpha(x_a - x_e)(x_a - x_f) - x_a^2\beta \\ * & \beta \end{smallmatrix} \right\} \\ &\cong \left\{ \begin{smallmatrix} * & x_a - x_d \\ * & x_b x_c - x_e x_f \end{smallmatrix} \right\} \{2\} \oplus \left\{ \begin{smallmatrix} * & x_a x_b x_c - x_d x_e x_f \\ * & \beta \end{smallmatrix} \right\} \end{aligned}$$

It is easy to see that the first summand is isomorphic to $C_p(D'_{IIIb})\langle 1 \rangle \{3\}$. Denote the second summand by $\Upsilon\{3\}$, so that we have $C_p(D_{IIIa}) \cong C_p(D'_{IIIb})\langle 1 \rangle \oplus \Upsilon$. By [Lemma 4.1](#), reflection along the middle strand is given by the ring homomorphism $\bar{\cdot} : R \rightarrow R$, $\bar{x}_a = -x_c$, $\bar{x}_b = -x_b$, $\bar{x}_c = -x_a$, $\bar{x}_d = -x_f$, $\bar{x}_e = -x_e$ and $\bar{x}_f = -x_d$. Since $\bar{\Upsilon} \cong \Upsilon$ by [Theorem 5.3](#) under this isomorphism, we obtain that $C_p(D'_{IIIa}) \cong C_p(D_{IIIb})\langle 1 \rangle \oplus \Upsilon$, which implies claim (e).

(f) This follows immediately from (a) and (d). \square

We will collectively refer to diagrams of type D_r and D_s as *resolved crossings*.

Proof. (of Theorem 5.1) We will prove the theorem by repeatedly reducing $C_p(D)$ according to [Proposition 5.5](#). At each step, we get a complex of matrix factorizations whose underlying graded object is $\bigoplus_i C_p(D_i)$ for some collection of singular diagrams in braid form. Following Wu [28], we define a complexity function on singular braids by $i(D) = \sum_j i_j$ where j runs over all resolved crossings in the diagram and i_j is 1 for an oriented smoothing and one plus the number of strands to the left of the crossing for a singular crossing. We show that each step of the reduction process decreases either the maximum complexity of diagrams D_i or the number of diagrams that have maximum complexity. This reduction can be performed as long as the maximum complexity is greater than 1. The only connected diagrams of complexity 1 are D_r and D_s , so if the maximum complexity is 1, $C_p(D)$ is the direct sum of shifts of $C_p(D_r)$ and $C_p(D_s)$ and the Lemma follows. To perform the reduction, choose a diagram of maximum complexity. The Lemma below guarantees that either D_{IIIa} or one of the diagrams on the left hand side of [Proposition 5.5\(a\)-\(d\)](#) or (f) is a subdiagram of D . In the latter case we can simply replace the complex on the left hand side by the the one on the right-hand side; notice that this reduces the number of diagrams of this complexity. If there is a subdiagram of type D_{IIIa} , we are given a complex of the form

$$\dots C^{k-1} \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} C^k \oplus C_p(D_{IIIa}) \xrightarrow{(\gamma \ \delta)} C^{k+1} \dots,$$

which is (up to a grading shift) isomorphic in $K^b(hmf_w(R))$ to

$$\dots C^{k-1} \oplus C_p(D_{IIIb}) \xrightarrow{\begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}} C^k \oplus C_p(D_{IIIa}) \oplus C_p(D_{IIIb}) \xrightarrow{(\gamma \ \delta \ 0)} C^{k+1} \dots,$$

which is in turn isomorphic to

$$\dots C^{k-1} \oplus C_p(D_{IIIb}) \rightarrow C^k \oplus C_p(D'_{IIIa}) \oplus C_p(D'_{IIIb}) \rightarrow C^{k+1} \dots,$$

so we once again were able to reduce the number of diagrams of the given complexity. \square

Lemma 5.6. *If D is a connected singular (open) braid diagram of complexity greater than 1, then it contains at least one of the following subdiagrams:*

- (i) *A resolved crossing of type D_r or D_s in rightmost position which is the only resolved crossing in this column,*
- (ii) *a diagram D_r which has the property that D stays connected when D_r is removed,*
- (iii) *a diagram of type D_{II} , D_{III} or D_{IV} .*

Proof. We prove the lemma by induction on the braid index. If the braid index is 2 and $i(D) > 1$, then we either have a subdiagram of type D_r , which can be removed without disconnecting the diagram, or we have at least two subdiagrams of type D_s and none of type D_r , so we can find D_{II} as a subdiagram. If the braid index is greater than 2, we may assume that there are at least two resolved crossings in rightmost position. We may also assume that we have no subdiagrams of type D_r in rightmost position, since we could remove them without disconnecting the diagram. If two such singular crossings are adjacent, we have found D_{II} as a subdiagram. Otherwise choose the topmost two such singular crossings and apply the induction hypothesis to the part of the braid between those two singular crossings, giving us either a subdiagram of the required type inside this part of the braid or, potentially after performing an isotopy, a diagram of type D_{III} or D_{IV} . \square

6. MUTATION INVARIANCE OF THE INNER TANGLE

The following simple lemma is at the heart of the proof. We will use it to show that invariance under mutation is essentially a property of the category of matrix factorizations associated to 2-tangles. The functors \mathcal{F} and \mathcal{G} are necessary to account for grading shifts; we suggest that the reader think of them as identity functors and of f as a natural transformation in the center of the category.

Lemma 6.1. *Let \mathcal{C} be an additive category and let \mathcal{F} , \mathcal{G} and $\bar{\cdot}$ be additive endofunctors of \mathcal{C} , where $\bar{\cdot} : \mathcal{C} \rightarrow \mathcal{C}$ is required to be the identity on objects and an involution on morphisms. Furthermore, let f be a natural transformation from \mathcal{F} to \mathcal{G} and let $\partial : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(\mathcal{G}A, \mathcal{F}B)$ be an operation defined on the Hom-sets of \mathcal{C} with the following properties*

- (1) *∂ is \mathbb{Z} -linear, i.e. for $\phi, \psi \in \text{Hom}_{\mathcal{C}}(A, B)$, $\partial(\phi - \psi) = \partial\phi - \partial\psi$.*
- (2) *For $\phi \in \text{Hom}_{\mathcal{C}}(A, B)$, $\mathcal{G}(\phi - \bar{\phi}) = f_B \partial\phi$ and $\mathcal{F}(\phi - \bar{\phi}) = \partial\phi f_A$.*
- (3) *Composable morphisms $\phi \in \text{Hom}_{\mathcal{C}}(A, B)$ and $\psi \in \text{Hom}_{\mathcal{C}}(B, C)$ satisfy a perturbed Leibniz rule: $\partial(\psi\phi) = \partial\psi \mathcal{G}\phi + \mathcal{F}\psi \partial\phi = \partial\psi \mathcal{G}\bar{\phi} + \mathcal{F}\psi \partial\phi$.*

If C is a chain complex over \mathcal{C} with differential d , then f gives rise to a chain morphism $f_C : \mathcal{F}C \rightarrow \mathcal{G}C$. Let \bar{C} be the chain complex obtained by applying $\bar{\cdot}$ to the differential of C . Then the mapping cones $\text{Cone}(f_C)$ and $\text{Cone}(f_{\bar{C}})$ are isomorphic.

Proof. We adopt the following conventions for the mapping cone. Let $\varepsilon_C : C \rightarrow C$ be the identity in even homological heights and the negative of the identity in odd heights. Note that ε_C commutes with morphisms of even homological degree and anti-commutes with morphisms of odd degree. Then the mapping cone $\text{Cone}(f_C)$

is given by $\mathcal{F}C[-1] \oplus \mathcal{G}C$ with differential $\begin{pmatrix} \mathcal{F}d & \\ f_C \varepsilon_C & \mathcal{G}d \end{pmatrix}$. Since d here has degree 1, it is easy to check that this defines a differential.

We claim that the horizontal arrows in the diagram below define an isomorphism between $\text{Cone}(f_C)$ and $\text{Cone}(f_{\bar{C}})$.

$$\begin{array}{ccc} \mathcal{F}C[-1] \oplus \mathcal{G}C & \xrightarrow{\begin{pmatrix} I & \partial d \varepsilon_C \\ & I \end{pmatrix}} & \mathcal{F}C[-1] \oplus \mathcal{G}C \\ \begin{pmatrix} \mathcal{F}d & \\ f_C \varepsilon_C & \mathcal{G}d \end{pmatrix} \uparrow & & \uparrow \begin{pmatrix} \mathcal{F}\bar{d} & \\ f_{\bar{C}} \varepsilon_{\bar{C}} & \mathcal{G}\bar{d} \end{pmatrix} \\ \mathcal{F}C[-1] \oplus \mathcal{G}C & \xrightarrow{\begin{pmatrix} I & \partial d \varepsilon_C \\ & I \end{pmatrix}} & \mathcal{F}C[-1] \oplus \mathcal{G}C \end{array}$$

$\begin{pmatrix} I & \partial d \varepsilon \\ & I \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} I & -\partial d \varepsilon \\ & I \end{pmatrix}$, so it remains to check that it defines a chain morphism, i.e. that $\begin{pmatrix} \mathcal{F}\bar{d} & \\ f_{\bar{C}} \varepsilon_{\bar{C}} & \mathcal{G}\bar{d} \end{pmatrix} \begin{pmatrix} I & \partial d \varepsilon \\ & I \end{pmatrix} = \begin{pmatrix} I & \partial d \varepsilon \\ & I \end{pmatrix} \begin{pmatrix} \mathcal{F}d & \\ f_C \varepsilon_C & \mathcal{G}d \end{pmatrix}$. We have $\deg(\mathcal{F}d) = \deg(\mathcal{G}d) = 1$, $\deg(\partial d) = 1 + (-1) - 0 = 0$ and $\deg f = 0 + 0 - (-1) = 1$, so this follows from $\mathcal{F}\bar{d} = \mathcal{F}d - \partial d f = \mathcal{F}d + \partial d \varepsilon f$, from $\mathcal{F}\bar{d} \partial d \varepsilon = -\partial d \mathcal{G}d \varepsilon = \partial d \varepsilon \mathcal{G}d$ and from $f_{\bar{C}} \partial d \varepsilon + \mathcal{G}\bar{d} = f \partial d + \mathcal{G}d = \mathcal{G}d$, where the second identity follows from $0 = \partial(d^2) = \partial d \mathcal{G}d + \mathcal{F}\bar{d} \partial d$. \square

Lemma 6.2. *Let $R = \mathbb{Q}[x_a, x_b, x_c, x_d]/(x_a + x_b = x_c + x_d)$ and let $\bar{\cdot}$ be the ring homomorphism defined by $\bar{x}_a = -x_b$, $\bar{x}_b = -x_a$, $\bar{x}_c = -x_d$ and $\bar{x}_d = -x_c$, which induces an involution functor on hmf_2 . Let \mathcal{F} be the grading shift functor $\{2\}$ and \mathcal{G} be the identity functor. Then there is a differential ∂ on the morphism spaces of hmf_2 satisfying the hypothesis of the previous lemma.*

Proof. $\bar{\cdot} : R \rightarrow R$ is well-defined since $\overline{x_a + x_b} = -x_b - x_a = -x_d - x_c = \overline{x_c + x_d}$. R is isomorphic to the polynomial ring in x_a , x_b and x_c . Substituting $x_b = -x_a$ in any expression of the form $r - \bar{r}$, we obtain 0, hence $r - \bar{r}$ is divisible by $x_a + x_b$ and we may define ∂ on R by $\partial r = \frac{r - \bar{r}}{x_a + x_b}$. Viewing the ring R as an additive category with one element, it is straightforward to check that ∂ satisfies the hypothesis of Lemma 6.1.

The differential descends to a differential on hmf_2 . First note that objects in hmf_2 are direct sums of one-term Koszul factorizations $K(u; v)$ with potential $w = p(x_c) + p(x_d) - p(x_a) - p(x_b)$. It follows from the proof of the one-crossing case of Lemma 4.1 that $\partial w = 0$ and $\partial v = 0$ for the two choices of v , that is $v = x_c - x_a$ and $v = (x_c - x_a)(x_c - x_b)$. This implies $0 = \partial w = \partial u v + \bar{u} \partial v = \partial u v$, hence $\partial v = 0$ since R does not have zero divisors. We define the differential of a morphism of such matrix factorizations,

$$\begin{array}{ccc} R\{\frac{\deg v' - \deg u'}{2}\} & \xrightleftharpoons[u']{v'} & R \\ \uparrow y & & \uparrow z \\ R\{\frac{\deg v - \deg u}{2} + \deg z\} & \xrightleftharpoons[u]{v} & R\{\deg z\} \end{array}$$

to simply be

$$\begin{array}{ccc} R\{\frac{\deg v' - \deg u'}{2} + 2\} & \xrightleftharpoons[u']{v'} & R\{2\} \\ \partial y \uparrow & & \uparrow \partial z \\ R\{\frac{\deg v - \deg u}{2} + \deg z\} & \xrightleftharpoons[u]{v} & R\{\deg z\} \end{array}$$

This is a morphism of matrix factorizations since $\partial y u = \partial(yu) = \partial(u'z) = u' \partial z$ and $\partial z v = \partial(zv) = \partial(v'y) = v' \partial y$.

Since any null-homotopic morphism

$$\begin{array}{ccc} R\{\frac{\deg v' - \deg u'}{2}\} & \xrightleftharpoons[u']{v'} & R \\ u'h + kv \uparrow & & \uparrow hu + v'k \\ R\{\frac{\deg w}{2} + \deg h\} & \xrightleftharpoons[u]{v} & R\{\deg h + \deg u\} \end{array}$$

is sent to the null-homotopic morphism

$$\begin{array}{ccc} R\{\frac{\deg v' - \deg u'}{2} + 2\} & \xrightleftharpoons[u']{v'} & R\{2\} \\ u' \partial h + \partial k v \uparrow & & \uparrow \partial h u + v' \partial k \\ R\{\frac{\deg w}{2} + \deg h\} & \xrightleftharpoons[u]{v} & R\{\deg h + \deg u\} \end{array}$$

∂ descends to a differential on hmf_2 .

The natural transformation ϕ is given by

$$\begin{array}{ccc} R\{\frac{\deg v - \deg u}{2}\} & \xrightleftharpoons[u]{v} & R \\ x_a + x_b \uparrow & & \uparrow x_a + x_b \\ R\{\frac{\deg v - \deg u}{2} + 2\} & \xrightleftharpoons[u]{v} & R\{2\} \end{array}$$

Since we can view (representatives of) morphisms in hmf_2 as pairs of elements of R , the fact that R satisfies the hypothesis of [Lemma 6.1](#) implies that hmf_2 does as well. \square

7. PROOF OF THE MAIN THEOREM

Before we can finish the proof, we need to borrow another Lemma from [\[23\]](#).

Lemma 7.1. (*Lemma 5.16 in [\[23\]](#)*) *Let D be the diagram of a single crossing with endpoints as in [Figure 4\(a\)](#). Then the maps $x_b : C_p(D)\{2\} \rightarrow C_p(D)$ and $x_c : C_p(D)\{2\} \rightarrow C_p(D)$ are homotopic. Since $x_a + x_b = x_c + x_d$, this of course implies that x_a and x_d are homotopic as well.*

Proof. Let $d_+ : C_p(D_r) \rightarrow C_p(D_s)$ be the differential of a positive crossing and $d_- : C_p(D_s) \rightarrow C_p(D_r)$ be the differential of a negative crossing. Clearly, $d_- d_+ = x_c - x_b : C_p(D_r) \rightarrow C_p(D_r)$ and $d_+ d_- = x_c - x_b : C_p(D_s) \rightarrow C_p(D_s)$, so d_\mp is a null-homotopy for $x_c - x_b : C_p(D_\pm) \rightarrow C_p(D_\pm)$. We ignored q -gradings above, the reader can easily check that the proof applies in the graded setting as well. \square

We are now ready to prove [Theorem 1.1](#). Given a pair of mutants L_1 and L_2 , we may assume, by [Theorem 3.2](#), that the mutation is realized as a mutation of type R_z whose inner tangle diagram D is in braid form. By [Theorem 5.1](#), there is an object C in $K^b(hmf_2)$ such that $C_p(D) \cong C$ in $K^b(hmf_w(R))$. Applying the ring isomorphism $\bar{\cdot}$, we obtain an isomorphism $\overline{C_p(D)} \cong \bar{C}$, hence by [Lemma 4.1](#), $C_p(\bar{D}) \cong \bar{C}$. Applying [Lemma 6.1](#), we obtain that $\text{Cone}(x_a + x_b : C\{2\} \rightarrow C)$ is isomorphic in hmf_2 , and hence in $hmf_w(R)$ to $\text{Cone}(x_a + x_b : \bar{C}\{2\} \rightarrow \bar{C})$. Taking the tensor product over $\mathbb{Q}[x_b, x_c, x_d]$ with the complex associated to the outer tangle, we get an isomorphism

$$\text{Cone}(x_a + x_b : C_p(L_1^\circ)\{2\} \rightarrow C_p(L_1^\circ)) \cong \text{Cone}(x_a + x_b : C_p(L_2^\circ)\{2\} \rightarrow C_p(L_2^\circ))$$

by [Proposition 2.3](#), where L_1° and L_2° denote L_1 and L_2 cut open at a , respectively. Because we consider only positive mutation, x_a and x_b lie on the same component of both L_1 and L_2 , so x_a and x_b are homotopic by repeated application of [Lemma 7.1](#). Hence we get an isomorphism

$$\text{Cone}(2x_a : C_p(L_1^\circ)\{2\} \rightarrow C_p(L_1^\circ)) \cong \text{Cone}(2x_a : C_p(L_2^\circ)\{2\} \rightarrow C_p(L_2^\circ))$$

and thus

$$\text{Cone}(x_a : C_p(L_1^\circ)\{2\} \rightarrow C_p(L_1^\circ)) \cong \text{Cone}(x_a : C_p(L_2^\circ)\{2\} \rightarrow C_p(L_2^\circ)).$$

Tensoring with $K(p'(x_a); 0)$ we get $\text{Cone}(\hat{C}_p(L_1^\circ)) \cong \text{Cone}(\hat{C}_p(L_2^\circ))$, which implies that the reduced homologies of L_1 and L_2 are isomorphic. \square

Proof (of [Corollary 1.2](#)). This follows directly from Theorem 1 in [\[23\]](#), which asserts that for sufficiently large n , the $\mathfrak{sl}(n)$ homology of a knot is a regraded version of its HOMFLY-PT homology. It is clear that we can recover the triple grading of HOMFLY-PT homology by choosing n large enough. \square

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